Introduction to the conformal net/vertex operator algebra correspondence

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Part 0: Overview

Axiomatizing CFT

- A (two-dimensional, chiral) conformal field theory comes with a lot of data: correlation functions, fusion products, braiding, characters, tensor categories, central charge, and much more.
- The goal is to provide axioms for a *subset* of the data in such a way that:
  1) it is possible to recover the remaining data
  2) expected behavior can be rigorously proven
  3) all physically relevant models satisfy the axioms
- Even in low dimensions this is very difficult, but it also has a knack for producing interesting and broadly applicable mathematics.

Conformal nets vs VOAs

- Conformal nets and vertex operator algebras are two approaches to axiomatizing 2d chiral CFT.
- They axiomatize different subsets of the data, using different mathematical tools.
- When we compare the two descriptions we can physical ideas to relate different areas of mathematics. E.g.

Conjugate sectors $\overset{\text{CNs}}{\rightsquigarrow}$ Finite Jones index

Conjugate sectors $\underset{\text{VOAs}}{\rightsquigarrow}$ Invertible fusion matrix

- This brings us closer to a complete mathematical picture of CFT, and leads to new connections e.g. between subfactors, VOAs, and tensor categories.

The plan

- A quick tour of conformal nets and vertex operator algebras for non-experts
- A discussion of ongoing programs to go back and forth between the two
- Recent achievements and open problems
- Next talk: Connections with Segal (functorial) CFT

Part 1: Definitions/vacuum sector

Conformal nets

A conformal net $\mathcal{A}$ is

- a Hilbert space $\mathcal{H}_0$ and a unit vector $\Omega \in H_0$
- for every interval $I \subset S^1$, a von Neumann algebra $\mathcal{A}(I) \subset B(H_0)$

◦ a projective unitary representation U of $\text{Diff}_+(S^1)$ on H_0

such that

- if $I \subset J$ then $\mathcal{A}(I) \subset \mathcal{A}(J)$
- if $I \cap J = \emptyset$ then $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute

◦ $U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma(I))$

◦ if $\text{supp}(\gamma) \subset I$ then $U(\gamma) \in \mathcal{A}(I)$

◦ if γ extends holomorphically to the disk $\mathbb{D}$, then $U(\gamma)\Omega = \Omega$

◦ Ω is cyclic for the $\mathcal{A}(I)$

Versions of this definition first appear in Fredenhagen-Rehren-Schroer '92 and Gabbiani-Fröhlich '93, following Haag-Kastler '64.

Examples of conformal nets: WZW models

- G - compact simple simply connected Lie group
- LG - the loop group $C^\infty(S^1, G)$
- $\pi_{k,0}$ - the level k vacuum representation of $\widetilde{LG}$ for $k \in \mathbb{Z}_+$

WZW models are given by:

$$\mathcal{A}_{G,k}(I) = \mathrm{vNA}\left(\{\pi_{k,0}(f) \, : \, \mathrm{supp}(f) \subseteq I\}\right)$$

Digression: unitarity

- Conformal nets axiomatize *unitary* chiral conformal field theories.
- In these examples the space of states has an inner product compatible with the other data (i.e. the algebras $\mathcal{A}(I)$ are closed under taking adjoints)
- Not all chiral CFTs are unitary. We are looking at a proper subset of theories by comparing conformal nets to *unitary* VOAs.

Unitary vertex operator algebras

A unitary vertex operator algebra is given by

- a finite-dimensionally graded inner product space $V = \bigoplus_{n=0}^{\infty} V(n)$ and a unit vector $\Omega \in V(0)$
- for every $a \in V$ a formal distribution $Y(a,z) \in \mathrm{End}(V)[[z^{\pm 1}]]$
- ◦ a conformal vector $\nu \in V$

such that

- $(z-w)^N [Y(a,z), Y(b,w)] = 0$ for N sufficiently large
- ◦ $Y(a,z)\Omega|_{z=0} = a$ and $Y(\Omega,z) = \mathrm{id}_V$
- ◦ If $Y(\nu,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, then L_n give a representation of the Virasoro algebra.
- ◦ The grading on V is given by L_0 and $[L_{-1}, Y(a,z)] = \frac{d}{dz} Y(a,z)$
- ◦ $(z-w)^N [Y(a,z^{-1})^*, Y(b,w)] = 0$ for N sufficiently large

This version of the definition of unitarity first appeared in Carpi-Kawahigashi-Longo-Weiner '18.

Example of unitary VOAs: WZW models

- $\mathfrak{g}$ - compact simple Lie algebra
- $L\mathfrak{g}^0$ - the polynomial loop algebra $\mathfrak{g}[z^{\pm 1}] \subset C^\infty(S^1, \mathfrak{g})$
- $\pi_{k,0}$ - the level $k \in \mathbb{Z}_+$ vacuum representation of $\widetilde{L\mathfrak{g}^0}_{\mathbb{C}}$
- For $X \in \mathfrak{g}_{\mathbb{C}}$ set $X_n = \pi_{k,0}(Xz^n)$.

The vertex algebra $V_{\mathfrak{g},k}$ is generated by fields

$$Y(X_{-1}\Omega, z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}$$

Comparing WZW models

- Representation of $C^\infty(S^1, G)$ on H
- Representation $\mathfrak{g}[z^{\pm 1}] \subset C^\infty(S^1, \mathfrak{g})$ on $V \subset H$
- Goodman-Wallach '84: the representation of $\mathfrak{g}[z^{\pm 1}]$ extends to $C^\infty(S^1, \mathfrak{g})$ and then exponentiates to $C^\infty(S^1, G)$.
- In vertex algebra terms, $Xf(z)$ acts by the *smeared field*

$$Y(X_{-1}\Omega, f) := \frac{1}{2\pi i} \int_{S^1} Y(X_{-1}\Omega, z) f(z)\, dz = \sum_{n \in \mathbb{Z}} \hat{f}(n) X_n.$$

- So $\mathcal{A}_{\mathfrak{g},k}(I)$ is generated by $\{e^{Y(X_{-1}\Omega, f)} : \mathrm{supp}(f) \subset I, X \in \mathfrak{g}\}$.

Intermediate step: Wightman CFT

- The fields $Y(a,z)$ are a priori formal distributions

$$\Big(f \in \mathbb{C}[z^{\pm 1}]\Big) \mapsto \Big(Y(a,f) : V \to V\Big)$$

where $Y(a,f) = \frac{1}{2\pi i}\int_{S^1} Y(a,z)f(z)\,\frac{dz}{z^{1-d_a}} = \sum_{n\in\mathbb{Z}} \hat{f}(n)a_n$

- These can be upgraded to operator-valued distributions

$$\Big(f \in C^\infty(S^1)\Big) \mapsto \Big(Y(a,f) : D \to H_V\Big)$$

where H_V is the Hilbert space completion and D is a dense domain invariant under all $Y(b,f)$ (Raymond-Tanimoto-T, also partial result in Carpi-Kawahigashi-Longo-Weiner '18)

- $Y(a,f)$ and $Y(b,g)$ commute when f and g have disjoint support.

- In fact, unitary VOAs are equivalent to unitary Wightman CFTs satisfying a uniform order condition.

From VOAs to conformal nets

- Starting from a unitary VOA V, set

$$\mathcal{A}_V(I) = \mathrm{vNA}\left(\{Y(a,f) : a \in V, \mathrm{supp}(f) \subset I\}\right)$$

- The operators $Y(a,f)$ are unbounded. The algebras are generated using bounded measurable functions. E.g. if $Y(a,f)$ is self-adjoint, $\mathrm{vNA}(Y(a,f))$ contains $\{e^{itY(a,f)}\}$.
- A central technical challenge in algebraic QFT:

Problem

Show that $\mathcal{A}_V(I)$ and $\mathcal{A}_V(J)$ commute when I and J are disjoint

- If so, $\mathcal{A}_V$ is a conformal net (broad framework in CKLW '18, in the presence of polynomial energy bounds)
- Direct solution for WZW and Virasoro examples (via Glimm-Jaffe-Nelson and linear energy bounds) as well as e.g. W_3 (Carpi-Tanimoto-Weiner), tools can extend this to many more examples (CKLW, Gui)

From conformal nets to VOAs

- In the reverse direction, two methods of characterizing fields from conformal nets: one in CKLW (à la Fredenhagen-Jörß), one in Raymond-Tanimoto-T, but neither guarantees "enough" fields
- Henriques described our joint work in progress which constructs a unitary VOA from an arbitrary conformal net, and characterizes which VOAs correspond to nets.
- Conjecture: every unitary VOA generates a conformal net.
- A conformal nets provide added analytic control over its VOA, and the VOA provides access to CFT data that is not easily seen in the conformal net setting.

Part 2: Representations and tensor products

Representations of conformal nets

A representation of a conformal net is given by:

- a family of representations $\lambda_I : \mathcal{A}(I) \to \mathcal{B}(H_\lambda)$, compatible with inclusion of intervals

From this we extract a subfactor:

$$\underbrace{\lambda_{I'}\big(\mathcal{A}(I')\big)}_{N} \subseteq \underbrace{\lambda_I\big(\mathcal{A}(I)\big)'}_{M}$$

which has a Jones-Kosaki index $[M : N] =: \text{index}(\lambda)$.

Theorem (Jones '83)

The set of possible subfactor indices is

$$\{4\cos^2(\pi/n) \;:\; n = 3, 4, \ldots\} \cup [4, \infty].$$

Question

Which subfactors arise in this manner?

Rational conformal nets

Definition

A conformal net is called rational if it has finitely many iso classes of irreducible representations, each with finite index.

An apparently stricter definition first appeared in Kawahigashi-Longo-Müger '01, later simplified in Longo-Xu '04 and Morinelli-Tanimoto-Weiner '18.

Theorem (KLM '01 [+ LX '04 + MTW '18])

If $\mathcal{A}$ is a rational conformal net, then $\mathrm{Rep}(\mathcal{A})$ *is naturally a unitary modular tensor category.*

The rigidity of $\mathrm{Rep}(\mathcal{A})$ corresponds to finiteness of the index.

Representations of vertex operator algebras

A representation of a VOA is given by:

- a state-field map $Y^M : V \to \text{End}(M)[[z^{\pm 1}]]$

Definition

A VOA is called rational if it has finitely many iso classes of representations and representations satisfy an appropriate complete reducibility property.

- The work of Huang/Huang-Lepowsky shows that under mild hypotheses the category $\text{Rep}(V)$ is a modular tensor category.
- Rigidity is not built into the definition of rationality.

Comparing modules

- If $\mathcal{A}$ comes from a VOA V, and M is a V-module, then the corresponding representation π^M of $\mathcal{A}$ is characterized by

$$\pi_I^M(Y(a,f)) = Y^M(a,f)$$

  when $\mathrm{supp}(f) \subset I$ (up to closure/extension).

- Conjecture that such a π^M exists (at least if M is "not too big")
- All conformal net modules should be of the form π^M, similar to Henriques' talk.

Comparing intertwiners

- If $\mathcal{Y} \in \binom{M}{K\,N}$ is an intertwining operator and $\mathrm{supp}(f) \subset I$, then $\mathcal{Y}(a, f)$ should intertwine $\pi_{I'}^{M}$ and $\pi_{I'}^{N}$.
- This is a fundamental link between the two tensor product theories.
- There are again deep technical challenges because the operator $\mathcal{Y}(a, f)$ is unbounded. These have been addressed in special cases in work by Gui and T.

Comparison: tensor categories

For conformal nets:

- The tensor product ("Connes' fusion") of two representations is given by an explicit construction, depending on a choice of interval I
- The construction is manifestly unitary, producing a unitary tensor category

For unitary VOAs:

- The tensor product of VOA modules is given by a universal property, depending on a choice of point $z \in \mathbb{C}$.
- Construction of a tensor category relies on a particular construction of the tensor product via Huang-Lepowsky theory.
- Have to select an appropriate category of modules.
- The tensor product module does not come with an inner product.
- Problem: describe the VOA module which corresponds to Connes' fusion.

Part 3: Applications (and advertising)

Application: positivity conjectures

Using the dictionary VOA $\longleftrightarrow$ CN, we translate the Connes' fusion inner product into VOA language:

Conjecture (Positivity conjecture)

The form on $M \otimes N$ given by

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{\boxtimes, z} := \left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N$$

is positive semidefinite.

- where M and N are unitary V-modules, $0 < |z| < 1$,
- $\mathcal{Y} \in \binom{V}{M^\dagger \, M}$, where $M^\dagger$ is the complex conjugate module,
- and $a \mapsto \tilde{a}$ is a certain explicit involution.

Conjecture (Strong positivity conjecture)

There is a canonical unitary V-module structure on a dense subspace of $\overline{M \otimes N}^{\langle , \rangle_\boxtimes}$ and an intertwining operator $\mathcal{Y}_\boxtimes$ such that $\mathcal{Y}_\boxtimes(a, z)b$ agrees with the 'identity' $M \otimes N \to \overline{M \otimes N}^{\langle , \rangle_\boxtimes}$.

For the appropriate category of modules/choice of intertwiners, this should be a tensor product.

Application: unitarity of VOA tensor categories

- Gui has shown that when the VOA is rational and the positivity conjecture holds, $\mathrm{Rep}^u(V)$ has a natural unitary structure
- Can verify positivity in examples by leveraging 'automatic positivity' for the corresponding conformal net, and solving Wassermann's transport equation:

$$\pi^N(\mathcal{Y}_M^+(b,g)^*\mathcal{Y}_M^+(a,f)) = \mathcal{Y}_\boxtimes(b,g)^*\mathcal{Y}_\boxtimes(a,f)$$

where $\mathcal{Y}_M^+ \in \binom{M}{M\ V}$ and $\mathcal{Y}_\boxtimes \in \binom{\square}{M\ N}$.

- In papers of Gui (and T) this has been done for WZW models, discrete series type ADE W-algebras, lattice models, and more

Application: rationality of conformal nets

- It is difficult to show that CN representations have finite index (see Gabbiani-Fröhlich '93); the corresponding property for VOA modules is known by Huang's general theory.
- Wassermann '98 initiated the program of using fields to show that CN representations have finite index with type A WZW. Followed by Toledano Laredo '97 in type D. Field theoretic calculations done 'by hand.'
- Gui '18 used smeared intertwining operators and VOA theory to prove rationality of type CG nets.
- Geometric methods in T'19 used to show rationality of all WZW nets and discrete series ADE W-algebras.
- Gui '20 showed that CN and VOA rep categories are equivalent in all of these examples, and more.

Future directions

- General theory identifying $\mathrm{Rep}(\mathcal{A}_V)$ and $\mathrm{Rep}(V)$?
- Tensor product theory for very badly behaved unitary VOAs?
- Modular invariance of characters for conformal nets?
- Non-unitary analogs of conformal nets?
- Construction of Haagerup CFT?

Thank you!