TWO-INTERVAL SUBFACTORS AND THE LONGO-REHREN CONSTRUCTION

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Abstract. Vacuum sectors of algebraic conformal field theories on the circle satisfy Haag duality: $A(I) = A(I')'$. However if E is a union of disconnected intervals, we instead get an inclusion $A(E) \subsetneq A(E')'$ that measures the failure of Haag duality in this case. We will show that this inclusion is isomorphic to the Longo-Rehren extension of a local algebra $A(I)$ by the category of representations of the net (assuming that the net is rational).

1. Conformal nets

The main object of study in this talk will be nets of von Neumann algebras on the circle. That is, an assignment $I \mapsto \mathcal{A}(I)$ from (open, non-dense) intervals of the circle S^1 to von Neumann algebras acting on a fixed Hilbert space $\mathcal{H}_0$. We assume that the net satisfies

- isotony (preserves inclusions),
- locality ($[\mathcal{A}(I), \mathcal{A}(J)] = 0$ when $I \cap J = \emptyset$),
- vacuum condition (there exists a unit norm vector Ω that is cyclic and separating for each $\mathcal{A}(I)$)
- irreducibility ($\mathcal{A}(I)$ are factors iff vacuum vector is unique iff $\bigvee \mathcal{A}(I) = B(\mathcal{H}_0)$)
- (conformal?) symmetry

The factors $\mathcal{A}(I)$ will be of type III_1, and after we add more hypotheses they will be hyperfinite.

To simplify a few technicalities, we will assume diffeomorphism (or conformal) covariance. That is, there is a strongly continuous projective unitary representation U of $\mathrm{Diff}(S^1)$ such that

- $U_g \mathcal{A}(I) U_g^* = A(g \cdot I)$
- If the support of g is contained in an interval I, then $U_g x U_g^* = x$ for all x in $A(J)$ with $I \cap J = \emptyset$.
- The vacuum vector Ω is invariant under the Mobius subgroup of $\mathrm{Diff}(S^1)$.
- The generator of the rotation subgroup $R_\theta \subset \mathrm{Diff}(S^1)$ is positive.

Example: $\mathcal{A}(I)$ is given by the vacuum positive energy representations of $LSU(n)$ at levels $k = 1, 2, \ldots$.

Since we're in subfactor seminar, we're looking to get subfactors out of nets. Locality says that if I take I, J with $I \cap J = \emptyset$, then $\mathcal{A}(I) \subset \mathcal{A}(J)'$ is a subfactor. However, the relatively commutant $\mathcal{A}(I)' \cap A(J)'$ will include $\mathcal{A}(K)$ for any interval K such that $I \subset K \subset J'$ (with J' the complementary open interval $S^1 \setminus \overline{J}$). That is, we can only hope for finite index subfactors when looking that $\mathcal{A}(I) \subset \mathcal{A}(I')'$. However, this does not work either.

- Haag duality: $\mathcal{A}(I')' = \mathcal{A}(I)$
- Geometric modular group: the modular conjugation of $A(I)$ acts by reflection, and the modular group acts by dilation.

However, if we cut the circle into four pieces, we will often get a non-trivial inclusion $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ (henceforth $\mathcal{A}(E) \subseteq \mathcal{A}(E')'$). We will sometimes also cut the circle at $e^{3\pi i/4}$ to get the real line picture: *figure*. The main implication of the "geometric modular group" is that if J is the modular conjugation for $\mathcal{A}(0, \infty)$, then $J\mathcal{A}(I)J = \mathcal{A}(-I)$. In fact, this "modular PCT symmetry" is sufficient - diffeomorphism covariance is asking a whole lot more. Possible hypotheses

- rational (finitely many irreps, finite index)
- split
- strong additivity
- finite index $\mathcal{A}(E) \subseteq \mathcal{A}(E')'$
- modular PCT symmetry

or

- rational
- split
- diffeomorphism covariant

Our goal today is to show that $\mathcal{A}(E) \subseteq \mathcal{A}(E')'$ is isomorphic to the Longo-Rehren extension of $\mathcal{A}(I_1)$ by the category of representations of $\mathcal{A}$.

2. Representation theory and superselection sectors

A (DHR) representation $\pi : \mathcal{A} \to B(\mathcal{H}_\pi)$ is a collection of (automatically normal) maps $\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)$ that satisfy isotony and diffeomorphism covariance. That is, there is a projective unitary representation V of $\mathrm{Diff}^\infty(S^1)$ on $\mathcal{H}_\pi$ such that $\pi_{gI} \circ \mathrm{Ad}\, U_g = \mathrm{Ad}\, V_g \circ \pi_I$. If π is irreducible, then V factors through a projective representation of $\mathrm{Diff}(S^1)$.

Examples are (non-vacuum) positive energy representations of $LSU(n)$.

These representations give us Jones-Wassermann subfactors $\pi_I(\mathcal{A}(I)) \subseteq \pi_{I'}(\mathcal{A}(I'))'$. These subfactors give us a lot of information about the representation, and are never trivial except in the vacuum representation. The standard approach in ACFT is to think of representations as localized endomorphisms.

We now fix an interval I_0.

Proposition. *If π is a DHR representation of a net $\mathcal{A}$, then π is unitarily equivalent to a representation ρ on $\mathcal{H}_0$ such that $\rho_{I_0'} = id$ and $\rho_J(\mathcal{A}(J)) \subseteq \mathcal{A}(J)$ whenever $I_0 \subseteq J$.*

Proof. Since all representations of a type III factor are equivalent, we may pick a unitary $w : \mathcal{H}_\pi \to \mathcal{H}_0$ such that $x = w\pi_{I_0'}(x)w^*$ for all $x \in \mathcal{A}(I_0')$. Now define a representation ρ of $\mathcal{A}$ on the vacuum Hilbert space by $\rho_K(x) = w\pi_K(x)w^*$. By construction, we have $\rho_{I_0'} = id$ so it just remains to show that ρ_J acts as an endomorphism of $\mathcal{A}(J)$ when $I_0 \subseteq J$.

First we show that

$$\mathcal{A}(J') = \bigvee_{\tilde{J} \Subset J'} \mathcal{A}(\tilde{J}),$$

where $\tilde{J} \Subset J'$ means that $\overline{\tilde{J}} \subset J'$. Take a sequence $g_n \in \mathrm{Diff}(S^1)$ (in the Mobius subgroup, if desired) such that $g_n \to 1$ and $g_n \cdot J \Subset J$. If $x \in \mathcal{A}(J)$, then $\mathrm{Ad}\, g_n x$ is a sequence in $\bigvee \mathcal{A}(\tilde{J})$ that converges strongly to x.

Thus it suffices to show that $\rho_J(\mathcal{A}(J))$ commutes with $\mathcal{A}(\tilde{J})$ for $\tilde{J} \Subset J'$. Choose an interval K such that $J \cup \tilde{J} \subset K$. For $x \in \mathcal{A}(J)$ and $\tilde{y} \in \mathcal{A}(\tilde{J})$ we have

$$\begin{aligned} \rho_J(x)\tilde{y} &= \rho_K(x\tilde{y}) \\ &= \rho_K(\tilde{y}x) \\ &= \tilde{y}\rho_J(x) \end{aligned}$$

Since the $\mathcal{A}(\tilde{J})$ generate $\mathcal{A}(J')$, we have that $x \in \mathcal{A}(J')' = \mathcal{A}(J)$, as desired. □

One then defines the dimension of a representation by $\dim \rho = [\mathcal{A}(I_0) : \rho(\mathcal{A}(I_0))]^{\frac{1}{2}}$.

Fixing $M = \mathcal{A}(I_0)$, we now have a functor of C^* tensor categories: unitary equivalence classes of representations $\longrightarrow \mathrm{Sect}(M)$. The objects in the image of this functor are called "superselection sectors" (or just "sectors" when the meaning is clear from context).

A net is called "rational" if the category of superselection sectors has finitely many irreducible objects, all with finite dimension. Such nets were first studied in this setting by Kawahigashi, Longo and Mueger under the name "completely rational," with the additional hypotheses of strong additivity (irrelevance of points) and finite μ-index (index of the two-interval inclusion in the vacuum sector). They showed that the natural braiding on superselection sectors is non-degenerate, and Kawahigashi and Longo later classified completely rational nets with central charge less than 1. Longo and Xu showed that diffeomorphism covariant rational nets are completely rational using an orbifold trick.

3. The two-interval and Longo-Rehren inclusions

We begin by describing the Longo-Rehren inclusion. If M is a type III factor and $\{\rho_i\}$ is a finite system of irreducible endomorphisms, then one can show that

$$\bigoplus_i \rho_i \otimes \rho_i^{op}$$

is the canonical endomorphism of some subfactor $N \subset M \otimes M^{op}$. Equivalently, this is the dual canonical endomorphism of some extension of M, which will be the basic construction of $N \subset M$. This is the infinite equivalent of the asymptotic inclusion for finite-depth subfactors.

To get a conformal net version of this construction, cut the circle into four intervals $I_1, \ldots, I_4$, and put $E = I_1 \cup I_3$ (so that $E' = I_2 \cup I_4$). The two-interval inclusion is $\mathcal{A}(E) \subseteq \mathcal{A}(E')'$. We will identify $\mathcal{A}(E) \subset \mathcal{A}(E')'$ with the Longo-Rehren construction applied to $\mathcal{A}(I_1)$ and the collection of all irreducible sectors of $\mathcal{A}$ (localized in I_1).

We proceed assuming the split property: that is, if I_1 and I_3 are disjoint intervals then $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \cong \mathcal{A}(I_1) \otimes \mathcal{A}(I_3)$. Let J be the modular conjugation of the interval from 1 to $e^{3\pi i/4}$. With $j = \operatorname{Ad} J$, we have an identification of $\mathcal{A}(I_3)$ with $\mathcal{A}(I_1)^{op}$ and $j\rho j = \rho^{op}$. If γ_E is the canonical endomorphism of this inclusion (w/r/t to the vacuum state), and θ_E is its restriction to $\mathcal{A}(E)$, then our main result is that

$$\theta_E = \bigoplus \rho_i \otimes j\rho_i j,$$

as a sector of $\mathcal{A}(E)$.

The outline of the proof is as follows.

- We will show that $\langle \rho_i \otimes j\rho_k j, \theta_E \rangle = \delta_{i,k}$.
- We will show that θ_E extends to a representation of the net $\mathcal{A} \otimes \mathcal{A}^{op}$.
- We will "show" that the only irreducible representations of $\mathcal{A} \otimes \mathcal{A}^{op}$ are of the form $\rho_i \otimes \rho_k^{op}$.

4. The proof

4.1. Step 1.

Proposition. *Let $N \subset M$ be a finite-index inclusion of type III factors. Let γ be a canonical endomorphism, and θ its restriction to N. For $\rho \in \operatorname{End}(N)$, there is an anti-isomorphism between the linear space of intertwiners $\operatorname{Hom}_N(\rho, \theta)$ and $\operatorname{Hom}_M(1, \rho)$ (that is, elements of M that intertwine the actions of id and ρ on N).*

Proof. Let ϵ be the minimal conditional expectation from M to N and let $v \in \operatorname{Hom}_M(1, \gamma)$ be the "Pimsner-Popa isometry". That is,

$$[M : N]\epsilon(xv^*)v = x$$

for all $x \in M$ (so that $M = Nv$). It is straightforward to check that the desired anti-isomorphism is given by

$$\begin{aligned} w \in \operatorname{Hom}_N(\rho, \theta) &\mapsto w^* v \in \operatorname{Hom}_M(1, \rho), \\ \psi \in \operatorname{Hom}_M(1, \rho) &\mapsto [M : N]\epsilon(v\psi^*) \in \operatorname{Hom}_N(\rho, \theta). \end{aligned}$$

□

Theorem. *Let ρ be an endomorphism of $\mathcal{A}$ localized in I_1, and σ be an endomorphism of $\mathcal{A}$ localized in I_2. Then $\rho\sigma |_{\mathcal{A}(E)} \prec \theta_E$ as a sector of $\mathcal{A}(E)$*

if and only if $[\sigma] = [\overline{\rho}]$ *as sectors of* $\mathcal{A}(I_1 \cup \overline{I_2} \cup I_3)$. *If such a subsector occurs, it occurs with multiplicity one.*

Proof. Suppose $\rho\sigma\mid_{\mathcal{A}(E)} \prec \theta_E$. Then there is a $\psi \in \mathcal{A}(E')'$ such that $\psi x = x\rho\sigma(x)$ for all $x \in \mathcal{A}(E)$. We also have $\psi x = x\psi = \rho\sigma(x)\psi$ for $x \in \mathcal{A}(I_2)$. By strong additivity, $1 \prec \rho\sigma$ as sectors of $\mathcal{A}(I_1 \cup \overline{I_2} \cup I_3)$. Since ρ and σ commute we also have $1 \prec \sigma\rho$, and $\sigma = \overline{\rho}$.

The converse is similar. □

Theorem. *If* ρ *is a superselection sector of* $\mathcal{A}$ *localized in* I_1, *then* $[j\rho j] = [\overline{\rho}]$ *as sectors of* $\mathcal{A}(I_1 \cup \overline{I_2} \cup I_3)$

Proof. Let $P = \mathcal{A}(0, \infty)$ in the line picture $(\mathcal{A}(1, e^{3\pi i/4}))$, and let w be the isometry standard implementation of ρ as an endomorphism of P (so that $j(w) = w$). For $x \in P$, we have $wx = \rho(x)w = \rho(j\rho j(x))w$. We also have

$$wj(x) = j(wx) = j(\rho(x)w) = (j\rho j)(j(x))w = \rho(j\rho j(j(x)))w.$$

Since $j(x)$ is an arbitrary element of P', we have $\rho(j\rho j(x))w = wx$ for all x in P or P', and hence in any local algebra by strong additivity.

Thus for $x \in \mathcal{A}(I_4)$ we have $[w, x] = 0$ so that $w \in \mathcal{A}(I_1 \cup \overline{I_2} \cup I_3)$. Since w intertwines 1 and $\rho \circ j\rho j$ on this algebra, we have that $\overline{\rho} = j\rho j$. □

Corollary. *Using the tensor product language, we have shown that*

$$\bigoplus \rho \otimes j\rho j \prec \theta_E$$

and no other $\rho_i \otimes j\rho_k j$ *occurs as a subsector.*

4.2. **Step 2.**

Theorem. *There is a superselection sector* η *of* $\mathcal{A} \otimes \mathcal{A}^{op}$ *such that* $\eta_{I_1} = \theta_E$.

Proof. We will use two standard facts from nets of subfactors. The first is that assuming strong additivity, localized endomorphisms of the restriction of a net to the line extend back to the circle.

Now define $\eta_{I_1} := \theta_E$, as a sector of $\tilde{\mathcal{A}}(I_1) := \mathcal{A}(I_1) \otimes \mathcal{A}(I_1)^{op}$. There is a basic result in nets of subfactors that says that if $I_1 \subseteq I$, there is an extension of η_{I_1} to I so that if $K \subseteq I \setminus I_1$ then $\eta_K = id$. This can be defined by the formula $v_1 x = \eta_I(x) v_1$, where v_1 is the Pimsner-Popa isometry for $A(E')'$ inside its basic construction with respect to $\mathcal{A}(E)$.

Fix an interval $I \supset I_1$ such that $I \backslash I_1$ is to the right of I_1. Choose a unitary u that transports η_I to an interval to the right of I. Then $\eta_I = \operatorname{Ad} u^*$, and this gives an extension on $\mathcal{A}(-\infty, c)$ where c is the right endpoint of I. The inductive limit of this extension gives the desired superselection sector. □

4.3. **Step 3.** We now discuss why irreps π of $\mathcal{A}_1 \otimes \mathcal{A}_2$ decompose as $\pi_1 \otimes \pi_2$, provided the $\mathcal{A}_i$ are completely rational. We will look at the "quasi-local" C^*-algebra (denoted by $C^*(\mathcal{A})$ or just $\mathcal{A}$), which is defined as the universal C^*-algebra generated by subalgebras $\mathcal{A}(I)$. A (covariant) representation of the net $\mathcal{A}$ is the same thing as a (covariant) representation of the C^*-algebra $\mathcal{A}$. A fundamental fact is that if ρ is an irreducible representation of a completely rational net, then $\rho(\mathcal{A})''$ is a factor of type I. We will sketchy why this is true.

For any pair of intervals $I \Subset \tilde{I}$, the split property says there is a factor of type I_∞ in $\mathcal{A}(\tilde{I}) \cap \mathcal{A}(I)$. In particular, we have a lot of copies of the compact operators sitting in $\mathcal{A}$. Restricting the representation to the subalgebra they generate, we get a direct integral decomposition over a set X. If X is not a singleton, this gives uncountably many mutually inequivalent representations of the compact subalgebra, which in turn (non-trivially) extends to uncountably many inequivalent representations of $\mathcal{A}$. This contradictions rationality.

Thus if π is an irrep of $\mathcal{A}_1 \otimes \mathcal{A}_2$, its restrictions to the tensor summands generate commuting type I factors, which give us a splitting of the representation.